Math 151
Lecture Notes
Section 2.1

Recall the formula for average velocity:

$$v_a = \frac{\Delta s}{\Delta t} = \frac{s_1 - s_0}{t_1 - t_0}$$

where s_1 and s_0 are the position coordinates of the object at times t_1 and t_0 respectively.

What about an instantaneous velocity? What if we wanted to know a velocity at one instant in time instead of an average over time? For example, how does the police officer give us a speeding ticket? The answer is to use the average velocity to approximate the instantaneous velocity, but to let the time interval between t_1 and t_0 be as small as possible.

Example:

1. Suppose a ball is thrown vertically upward and the height in feet of the ball *t* seconds after its release is modeled by the function $s(t) = -16t^2 + 29t + 6$ for $0 \le t \le 2$. Approximate the instantaneous velocity at time $t = 0.25$ seconds.

t_1	v_a

Geometrically speaking, the instantaneous velocity is the slope between the points (t_0, s_0) and (t_1, s_1). We hold (t_0, s_0) fixed and let (t_1, s_1) move ever closer to (t_0, s_0). As the points get infinitely close to each other the instantaneous velocity approximates what we could call the slope of the position curve at the point (t_0, s_0).

This idea of letting the independent variable get closer to a fixed value and measuring the value of the dependent variable as this happens is the concept of the limit. We would write the velocity problem as follows

$$\lim_{t \to 0.25} \frac{-16t^2 + 29t - 6.25}{t - 0.25}$$

Theorem 2.1.1 Limits (Informally): If the values of $f(x)$ can be made as close as we like to *L* by taking values of *x* sufficiently close to *a* (but not equal to *a*), then we write:

$$\lim_{x \to a} f(x) = L$$

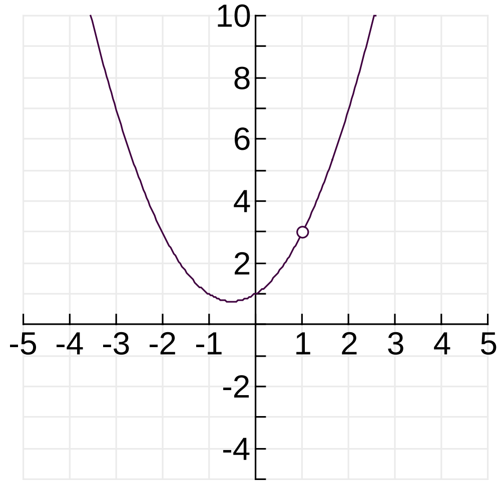
Which is read "the limit of $f(x)$ as *x* approaches *a* is *L*"

Sometimes we write the above as: "$f(x) \to L$ as $t \to a$"

Examples: Evaluate the limits numerically and graphically.

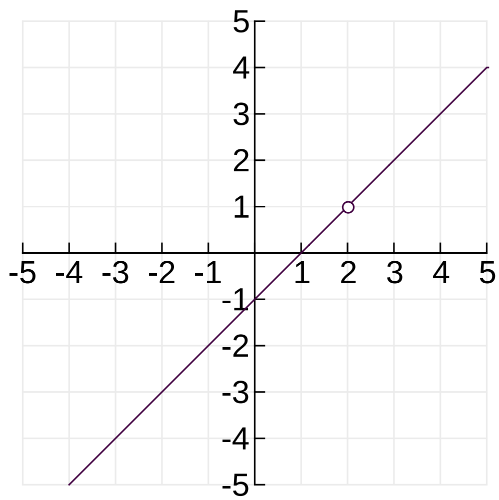
2. $f(x) = \dfrac{x^3 - 1}{x - 1} \quad x \neq 1$

x	*f*(*x*)



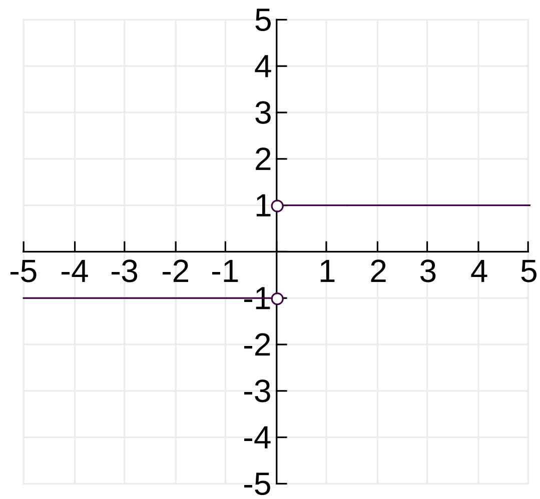
3. $\lim_{x \to 2} \dfrac{x^2 - 3x + 2}{x - 2}$

x	*f*(*x*)



4. $\lim_{x \to 0} \frac{|x|}{x}$

x	$f(x)$



Theorem 2.1.2 One-Sided Limits (Informally): If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but greater than a), then we write

$$\lim_{x \to a^+} f(x) = L$$

Which is read "the limit of $f(x)$ as x approaches a from the right is L." Similarly, If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but less than a), then we write

$$\lim_{x \to a^-} f(x) = L$$

Which is read "the limit of $f(x)$ as x approaches a from the left is L."

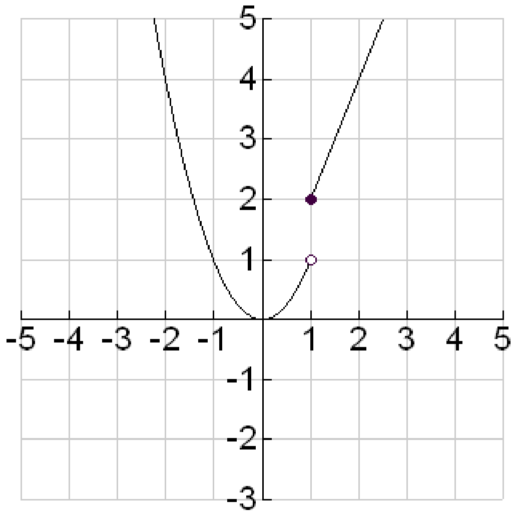
There is no guarantee that a limit will exist. If the values of $f(x)$ do not get closer to a single number L as $x \to a$ then we say the limit of $f(x)$ as x approaches a ***does not exist*** (as in example 4). The same can be said for one-sided limits. Let's take a closer look at example 4.

Theorem 2.1.3: The two-sided limit of a function $f(x)$ exists at a if and only if both of the one-sided limits exist at a and have the same value; that is

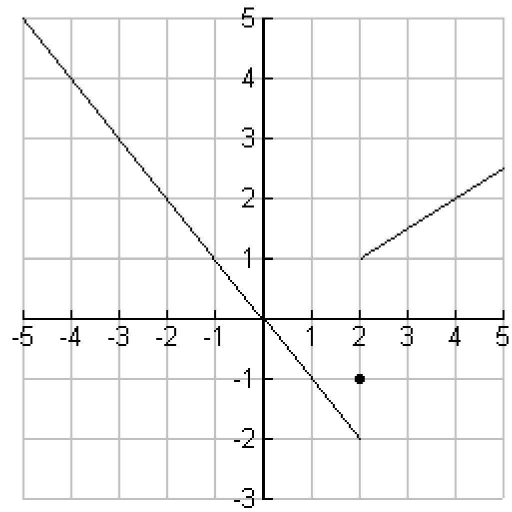
$$\lim_{x \to a} f(x) = L \text{ if and only if } \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$

Examples: Use the given graphs of $y = f(x)$ to approximate both of the one-sided limits and the two-sided limit at the given x value.

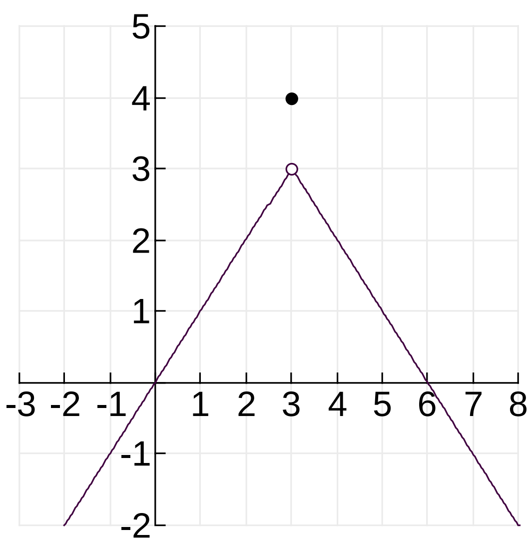
5.



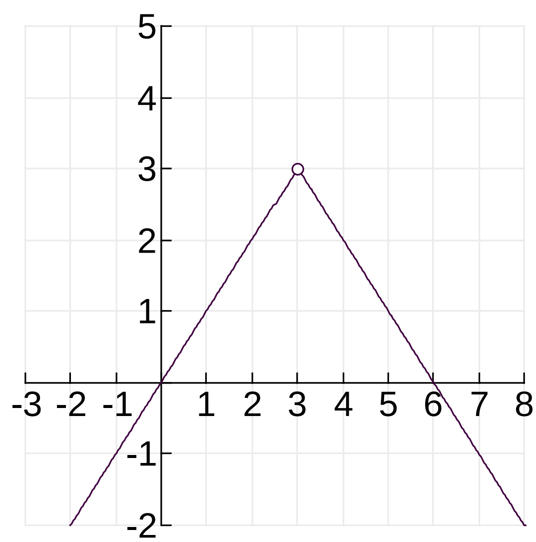
a) $x \to 1$



b) $x \to 2$



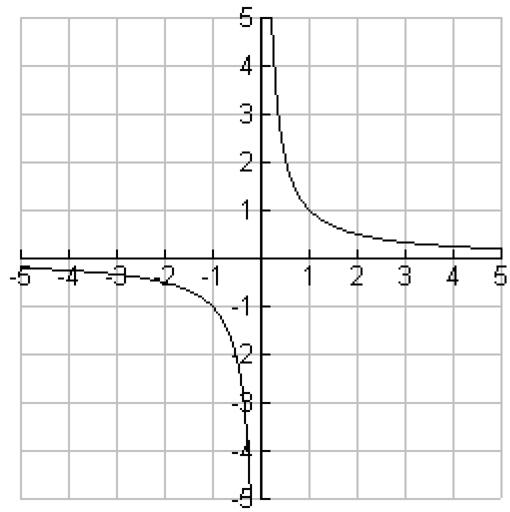
c) $x \to 3$



d) $x \to 3$

Consider the limits $\lim_{x \to 0^-} \frac{1}{x}$ and $\lim_{x \to 0^+} \frac{1}{x}$

x	$f(x)$



Technically these limits fail to exist because the values of $f(x)$ increase (or decrease) indefinitely at $x = 0$. However, the spirit of the limit is to describe the behavior of the function, and that we can do to the left and right of $x = 0$.

Theorem 2.1.4 Infinite Limits (Informally): If the values of $f(x)$ increase indefinitely as *x* approaches *a* from the right of left, then we write

$$\lim_{x \to a^+} f(x) = \infty \text{ or } \lim_{x \to a^-} f(x) = \infty$$

as appropriate, and we say that $f(x)$ ***increases without bound***, or $f(x) \to \infty$ as $x \to a^+$ or $x \to a^-$. Similarly, If the values of $f(x)$ decrease indefinitely as *x* approaches *a* from the right of left, then we write

$$\lim_{x \to a^+} f(x) = -\infty \text{ or } \lim_{x \to a^-} f(x) = -\infty$$

as appropriate, and we say that $f(x)$ ***decreases without bound***, or $f(x) \to \infty$ as $x \to a^+$ or $x \to a^-$. Moreover, if both one-sided limits are ∞ or both are $-\infty$ then we write

$$\lim_{x \to a} f(x) = \infty \text{ or } \lim_{x \to a} f(x) = -\infty \text{ respectively.}$$

Definition 2.1.5: A line $x = a$ is called a ***vertical asymptote*** of the graph of a function *f* if $f(x) \to \pm\infty$ as *x* approaches *a* from the left or right.

Example: Find the vertical asymptote of the function, if any exist.

6. $f(x) = \dfrac{2x^2 - 3x + 14}{x - 3}$

Theorem 2.1.6 Limits at Infinity (Informally): If the values of $f(x)$ eventually get closer and closer to a number L as x increases without bound, then we write:

$$\lim_{x\to\infty} f(x) = L \text{ or } f(x) \to L \text{ as } x \to \infty$$

Similarly, if the values of $f(x)$ eventually get closer and closer to a number L as x decreases without bound, then we write:

$$\lim_{x\to-\infty} f(x) = L \text{ or } f(x) \to L \text{ as } x \to -\infty$$

Definition 2.1.7: A line $y = L$ is called a ***horizontal asymptote*** of the graph of a function f if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$.

Example: Find the horizontal asymptote of the function $f(x) = \frac{3x^2 - 2x - 7}{x^2 - 1}$