

Math 151

Lecture Notes

Section 2.4

So far we have been discussing the concept of limit informally. We have said $f(x)$ “approaches” L as x “approaches” a . This has been good enough to develop an intuitive grasp of the concept of limit. However, mathematicians need something more precise. Something that can be used to demonstrate proof of the properties we have discussed so far.

Definition 2.4.1 Limit Definition: Let $f(x)$ be defined for all x in some open interval containing the number a , with the possible exception that $f(x)$ need not be defined at a . We will write

$$\lim_{x \rightarrow a} f(x) = L$$

if given any number $\varepsilon > 0$ we can find a number $\delta > 0$ such that $|f(x) - L| < \varepsilon$ if $0 < |x - a| < \delta$.

Examples: Given the value of the limit and a value for ε find the corresponding value for δ .

1. $\lim_{x \rightarrow 2} (2x + 9) = 13$ and $\varepsilon = 0.01$

2. $\lim_{x \rightarrow 3} (x^2) = 9$ and $\varepsilon = 0.05$

Examples: Use the definition to prove that the limit has the given value.

3. $\lim_{x \rightarrow 5} (x^2 - 16) = 9$

The above definition is great, but it only works for two-sided limits. What about all the other limits that we have considered? All we need to do is tweak the definition we already have and it will work for the other situations.

Definition 2.4.2: Let $f(x)$ be defined for all x in some infinite open interval extending in the positive x -direction. We will write

$$\lim_{x \rightarrow \infty} f(x) = L$$

if given any number $\varepsilon > 0$ there corresponds a positive number N such that $|f(x) - L| < \varepsilon$ if $x > N$.

Definition 2.4.3: Let $f(x)$ be defined for all x in some infinite open interval extending in the negative x -direction. We will write

$$\lim_{x \rightarrow -\infty} f(x) = L$$

if given any number $\varepsilon > 0$ there corresponds a negative number N such that $|f(x) - L| < \varepsilon$ if $x < N$.

Definition 2.4.4: Let $f(x)$ be defined for all x in some open interval containing the number a , with the possible exception that $f(x)$ need not be defined at a . We will write

$$\lim_{x \rightarrow a} f(x) = \infty$$

if given any positive number M we can find a number $\delta > 0$ such that $f(x) > M$ if $0 < |x - a| < \delta$.

Definition 2.4.5: Let $f(x)$ be defined for all x in some open interval containing the number a , with the possible exception that $f(x)$ need not be defined at a . We will write

$$\lim_{x \rightarrow a} f(x) = -\infty$$

if given any negative number M we can find a number $\delta > 0$ such that $f(x) < M$ if $0 < |x - a| < \delta$.

The definitions of one-sided limits are similar. All we have to do for a limit from the left is to change the condition $0 < |x - a| < \delta$ to $a - \delta < x < a$. Similarly for a limit from the right all we have to do is change $0 < |x - a| < \delta$ to $a < x < a + \delta$.

Example:

4. Prove that $\lim_{x \rightarrow -\infty} \frac{1}{x} = 0$